Abstract

Recent theoretical results establish that time-consistent valuations (i.e. pricing operators) can be created by backward iteration of one-period valuations. In this paper we investigate the continuous-time limits of well-known actuarial premium principles when such backward iteration procedures are applied. We show that the iterated variance premium principle converges to the non-linear exponential indifference valuation. Furthermore, we show that the iterated standard-deviation principle converges to an expectation under an equivalent martingale measure and that the Cost-of-Capital principle, which is widely used by the insurance industry, converges to the same price as the standard-deviation principle. Finally, we study the converge of market-consistent extensions of these pricing principles.

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1 Introduction

Standard actuarial premium principles usually consider a static premium calculation problem: what is the price today of an insurance contract with payoff at time $T$. See, for example, the textbooks by Bühlmann (1970), Gerber (1979), or Kaas et al. (2008). Also, the study of risk measures, and the closely related concept of monetary risk measures has been studied in such a static setting. See, for example, Artzner et al. (1999), Cheridito et al. (2005). Also the study of utility indifference valuations has mainly confined itself to this static setting. For different applications we mention a few papers: Henderson (2002), Young and Zariphopoulou (2002), Hobson (2004), Musiela and Zariphopoulou (2004), Monoyios (2006), and the recent book by ?.

Financial pricing usually considers a “dynamic” pricing problem: how does the price evolve over time until the final payoff date $T$. This dynamic perspective is driven by the focus on hedging and replication. This literature started by the seminal paper of Black and Scholes (1973) and has been immensely generalised to broad classes of securities and stochastic processes, see Delbaen and Schachermayer (1994).

In recent years, researchers have begun to investigate risk measures in a dynamic setting, where the question of constructing time-consistent (or “dynamic”) risk-measures has been investigated. See, Riedel (2004), Cheridito et al. (2006), Roorda et al. (2005), Rosazza Gianin (2006), Artzner et al. (2007). In a recent paper Jobert and Rogers (2008) show how time-consistent valuations can be constructed via backward induction of static one-period risk-measures (or “valuations”).

Another branch of literature considers risk measures/valuations in a so-called market-consistent setting. This started by the pricing of contracts in an incomplete market setting, where one seeks to extend the arbitrage-free pricing operators (which are only defined in a complete market setting) to the larger space of (partially) unhedgeable contracts. The paper by Hodges and Neuberger (1989) is often cited as the root-idea of the utility-indifference pricing literature mentioned above. A related branch of literature extends the arbitrage-free pricing operators using (local) risk-minimisation techniques and the related notion of minimal martingale measures, see Föllmer and Schweizer (1989), Schweizer (1995), Delbaen and Schachermayer (1996). A rich duality theory has been developed that makes deep connections between utility maximisation and minimisation over martingale measures, see Cvitanic and Karatzas (1992), Kramkov and Schachermayer (1999), for a very elegant summary we refer to Rogers (2001).

Using utility-indifference (and duality) methods, the market-consistency of pricing operators is automatically induced. However, an explicit formal
definition of market-consistent pricing operators has only begun to emerge recently, see Kupper et al. (2008) and Malamud et al. (2008).

In this paper we want to investigate well-known actuarial premium principles such as the variance principle and the standard-deviation principle, and study their extension into both time-consistent and market-consistent directions. The method we use to construct these extensions is to first consider one-period valuations, then extend this to a multi-period setting using the backward iteration method of Jobert and Rogers (2008) for a given discrete time-step $\Delta t$, and finally consider the continuous-time limit for $\Delta t \to 0$. We show that the extended variance premium principle converges to the non-linear exponential indifference valuation. Furthermore, we show that the extended standard-deviation principle converges to an expectation under an equivalent martingale measure. Finally, we show that the Cost-of-Capital principle, which is widely used by the insurance industry, converges to the same limit as the standard-deviation principle.

The rest of this paper is organised as follows. We start in Section 2 where we focus initially on the case of pure insurance risk. This allows us to clearly demonstrate the construction we use to derive the continuous-time limit of time-consistent actuarial pricing principles. We derive the limit for the Variance principle, the Standard-Deviation principle and the Cost-of-Capital principle. In Section 3 we extend our setup and combine unhedgeable insurance risk with hedgeable financial risk, and derive the continuous-time limits of the actuarial pricing principles when these are both market-consistent and time-consistent. We summarise and conclude in Section 4.

2 Pure Insurance Risk

In this section we focus on the case of pure insurance risk. This enables us to clearly demonstrate the construction we use to derive the continuous-time limit of time-consistent actuarial pricing principles. In Section 3, we will then include market-consistency into the picture.

2.1 Diffusion Setting

We start by considering an unhedgeable insurance process $y_t$, which is given by a diffusion equation:

$$dy = a(t, y) dt + b(t, y) dW.$$  \hspace{1cm} (2.1)
We also consider a discretisation scheme for the insurance process in the form of a binomial tree:

$$y(t + \Delta t) = y(t) + a\Delta t + \begin{cases} 
+b\sqrt{\Delta t} & \text{with prob. } \frac{1}{2} \\
-b\sqrt{\Delta t} & \text{with prob. } \frac{1}{2}
\end{cases}$$ (2.2)

where we have suppressed the dependence of $a$ and $b$ on $(t, y)$ to lighten the notation. Note that we have restricted ourselves to a Markovian diffusion setting, which allows us to give a very simple mathematical derivation of our results. In Section 2.2.5 we discuss alternatives for this restrictive assumption.

Given the discrete-time setting (2.2), we can now create time-consistent pricing operators ("valuations"), using the backward induction method of Jobert and Rogers (2008). Let us denote a one-step valuation by $\Pi[]$, and the resulting price at $(t, y)$ by $\pi(t, y)$:

$$\pi(t, y) = \Pi_t[\pi(t + \Delta t, y(t + \Delta t))]$$ (2.3)

In words: the price $\pi(t, y)$ is obtained by applying at time $t$ the one-step valuation $\Pi_t[]$ to the random variable $\pi(t + \Delta t, y(t + \Delta t))$, which is the price obtained in the previous time-step.

### 2.2 Variance Principle

If we consider an insurance contract with a payoff at time $T$ defined as a function $f(y(T))$, then the actuarial Variance Principle $\Pi_v[]$ is defined as (see, e.g. Kaas et al., 2008)

$$\Pi_v[f(y(T))] = E_t[f(y(T))] + \frac{1}{2} \alpha \text{Var}_t[f(y(T))],$$ (2.4)

where $E_t[]$ and $\text{Var}_t[]$ denote the expectation and variance operators conditional on the information available at time $t$ under the “real-world” probability measure $P$.

Note that in the standard actuarial literature (see, e.g. Kaas et al., 2008), discounting is usually ignored. To facilitate the discussion, we will first derive the continuous-time limit of the variance principle without using discounting in Section 2.2.1. We will then consider case with discounting in Section 2.2.2, and discuss further generalisations in Section 2.2.5.

#### 2.2.1 No Discounting

In the binomial tree discretisation we can obtain an explicit expression for a one-step variance price $\pi^v(t, y)$ by substituting (2.4) into (2.3):

$$\pi^v(t, y(t)) = E_t[\pi^v(t + \Delta t, y(t + \Delta t))] + \frac{1}{2} \alpha \text{Var}_t[\pi^v(t + \Delta t, y(t + \Delta t))].$$ (2.5)
We are now interested in considering the limit for $\Delta t \downarrow 0$. We assume that $\pi^\gamma(t + \Delta t, y)\,\equiv\, \pi^\gamma(t,y(t) + h)$ is sufficiently smooth to be twice continuously differentiable in $y$, such that we can apply for all values of $y$ the Taylor approximation

$$
\pi^\gamma(t + \Delta t, y(t) + h) = \pi^\gamma(t + \Delta t, y(t)) + 
\pi^\gamma_y(t + \Delta t, y(t)) h + \frac{1}{2} \pi^\gamma_{yy}(t + \Delta t, y(t)) h^2 + O(h^3), \quad (2.6)
$$

where subscripts on $\pi^\gamma$ denote partial derivatives. If we substitute this Taylor approximation for the binomial approximation (2.2) into (2.5) and gather all terms in ascending orders of $\Delta t$, we obtain

$$
\pi^\gamma(t, y(t)) - \pi^\gamma(t + \Delta t, y(t)) = a \pi^\gamma_y(t + \Delta t, y(t)) \Delta t + \frac{1}{2} b^2 \pi^\gamma_{yy}(t + \Delta t, y(t)) \Delta t + \frac{1}{2} \alpha \left( b \pi^\gamma_y(t + \Delta t, y(t)) \right)^2 \Delta t + O(\Delta t^2). \quad (2.7)
$$

If we divide by $\Delta t$ we obtain

$$
\frac{\pi^\gamma(t, y(t)) - \pi^\gamma(t + \Delta t, y(t))}{\Delta t} = a \pi^\gamma_y(t + \Delta t, y(t)) + \frac{1}{2} b^2 \pi^\gamma_{yy}(t + \Delta t, y(t)) + \frac{1}{2} \alpha \left( b \pi^\gamma_y(t + \Delta t, y(t)) \right)^2 + O(\Delta t). \quad (2.8)
$$

We can now take the limit for $\Delta t \downarrow 0$. The left-hand side of (2.8) converges to (minus) the partial derivative of $\pi^\gamma()$ with respect to $t$ (i.e. $-\pi_t^\gamma(t, y(t))$), and we obtain

$$
\pi_t^\gamma + a \pi_y^\gamma + \frac{1}{2} b^2 \pi_{yy}^\gamma + \frac{1}{2} \alpha (b \pi_y^\gamma)^2 = 0, \quad (2.9)
$$

where we have suppressed (again) the dependence on $t$ and $y$ to lighten the notation.

Note, that equation (2.9) is a non-linear partial differential equation that describes the behaviour of the variance price $\pi^\gamma(t, y)$ as a function of $t$ and $y$. The pde is subject to the boundary condition $\pi^\gamma(T, y(T)) = f(y(T))$ which is the payoff of the insurance contract at time $T$.

In general, the proof of the existence of solutions of non-linear pde’s can be very complicated, and we discuss this subject further in Section 2.2.5. Luckily, in this particular case, we can study the solution of (2.9) by employing a transformation of the solution that removes the non-linearity from the pde. Consider the auxiliary function $h^\gamma(t, y) := \exp\{\alpha \pi^\gamma(t, y)\}$. The original function $\pi^\gamma(t, y)$ can be obtained from the inverse relation $\pi^\gamma(t, y) = \frac{1}{\alpha} \ln h^\gamma(t, y)$. If we now apply the chain-rule of differentiation, we can express the partial derivatives of $\pi^\gamma()$ in terms of $h^\gamma()$ as

$$
\pi_t^\gamma = \frac{1}{\alpha} \frac{h_t^\gamma}{h^\gamma}, \quad \pi_y^\gamma = \frac{1}{\alpha} \frac{h_y^\gamma}{h^\gamma}, \quad \pi_{yy}^\gamma = \frac{1}{\alpha} \frac{h_y^\gamma h_y^\gamma - (h_y^\gamma)^2}{(h^\gamma)^2}. \quad (2.10)
$$

5
If we substitute these expressions into (2.9), the non-linear terms cancel and we obtain a linear pde for $h^\nu(t, y)$:

$$h^\nu_t + ah^\nu_y + \frac{1}{2}b^2 h^\nu_{yy} = 0. \quad (2.11)$$

Hence, by considering the transformed function $h^\nu(t, y)$ we have managed to obtain a linear pde for $h^\nu()$. The boundary condition at $T$ is given by

$$h^\nu(T, y(T)) = \exp\{\alpha \pi^\nu(T, y(T))\} = \exp\{\alpha f(y(T))\}. \quad (2.12)$$

Using the Feynman-Kač formula, we can express the solution of (2.11) as

$$h^\nu(t, y) = E_t \left[ e^{\alpha f(y(T))} \mid y(t) = y \right], \quad (2.13)$$

where the expectation is taken with respect to the stochastic process $y(t)$ defined in equation (2.1) conditional on the information that at time $t$ the process $y(t)$ is equal to $y$. From the representation (2.12) follows immediately that we can express $\pi^\nu(t, y)$ as

$$\pi^\nu(t, y) = \frac{1}{\alpha} \ln E_t \left[ e^{\alpha f(y(T))} \mid y(t) = y \right]. \quad (2.13)$$

Note that this representation of the variance-price $\pi^\nu()$ is identical to the exponential indifference price which has been extensively studied in recent years. See, for example Henderson (2002), Young and Zariphopoulou (2002) or Musiela and Zariphopoulou (2004). For a nice overview of recent advances in indifference pricing, we refer to the book by ?.

To summarise this section, we have established that the continuous-time limit of the iterated actuarial variance principle is the exponential indifference price.

### 2.2.2 Discounting

Up to now we have ignored discounting in our derivation. (Or equivalently, we assumed that the interest rate is equal to zero.) In a time-consistent setting, it is important to take discounting into consideration, as money today cannot be compared to money tomorrow.

If we consider the definition of the variance principle given in (2.4), it seems that we are adding apples and oranges together. The first term $E_t[ f(y(T)) ]$ is a quantity in monetary units (say $\mathbb{E}$) at time $T$. However, the second term $\text{Var}_t[ f(y(T)) ]$ is basically the expectation of $f(y(T))^2$, and is therefore a quantity in units of $(\mathbb{E})^2$. The way to rectify this situation is by understanding that the parameter $\alpha$ is not a dimensionless quantity, but is a quantity expressed in units of $1/\mathbb{E}$. This should not come as a surprise. The parameter $\alpha$ is in fact the absolute risk aversion parameter introduced by seminal paper...
by Pratt (1964) where he derives the variance principle as an approximation “in the small” of the price that an economic agent facing a decision under uncertainty should ask.

To stress in our notation the units in which the absolute risk aversion $\alpha$ is expressed, we will rewrite the absolute risk aversion as the relative risk aversion $\gamma$ (also introduced by Pratt, 1964), which is a dimensionless quantity, divided by a benchmark wealth-level $X(T)$, which is expressed in $\mathbf{E}$ at time $T$. If we now assume a constant rate of interest $r$, we can then set our benchmark wealth as $X(T) = X_0 e^{rt}$. Hence, we rewrite our variance principle as

$$\Pi_t[f(y(T))] = E_t[f(y(T))] + \frac{1}{2} \frac{\gamma}{X_0 e^{rt}} \text{Var}_t[f(y(T))]. \quad (2.14)$$

Note, that $\Pi_t[]$ leads to a “forward” price expressed in units of $\mathbf{E}$ at time $T$.

Given the enhanced definition (2.14) of the variance principle including discounting, we can now proceed as in Section 2.2.1. For a single binomial step, we obtain the following expression for the price:

$$\pi^\nu(t, y(t)) = e^{-r\Delta t}\left(E_t[\pi^\nu(t + \Delta t, y(t + \Delta t))] + \frac{1}{2} \frac{\gamma}{X_0 e^{rt(t+\Delta t)}} \text{Var}_t[\pi^\nu(t + \Delta t, y(t + \Delta t))]\right). \quad (2.15)$$

Note that we have included an additional discounting term $e^{-r\Delta t}$ to discount the values from time $t + \Delta t$ back to time $t$. Using a similar derivation as before, we arrive at the following partial differential equation for $\pi^\nu(t, y)$:

$$\pi^\nu_t + a\pi^\nu_y + \frac{1}{2} b^2 \pi^\nu_{yy} + \frac{1}{2} \frac{\gamma}{X_0 e^{rt}} (b\pi^\nu_y)^2 - r\pi^\nu = 0. \quad (2.16)$$

This non-linear PDE can again be linearised by considering the transformation $h^\nu(t, y) = \exp\left\{\frac{\gamma}{X_0 e^{rt}} \pi^\nu(t, y)\right\}$, which leads to the following expression for the solution of (2.16):

$$\pi^\nu(t, y) = \frac{X_0 e^{rt}}{\gamma} \ln E\left[e^{\frac{\gamma}{X_0 e^{rt}} f(y(T))}\bigg| y(t) = y\right]. \quad (2.17)$$

From this result we see that the discounting is incorporated into the non-linear pricing formula, by expressing all units relative to the “benchmark wealth” $X(t) = X_0 e^{rt}$.

\footnote{For general results concerning “benchmark pricing” in a linear setting we refer to Platen (2006) and the book by Platen and Heath (2006).}
2.2.3 Current price as benchmark

In the previous subsection we have taken the benchmark wealth to be a risk-free investment $X_0e^{rt}$. Another interesting example is when we consider the current price $\pi(t, y)$ as the benchmark wealth.

This then leads to a new pricing operator, which we will denote by $\pi^p()$. The one-step valuation is then given by

$$
\pi^p(t, y) = e^{-r\Delta t} \left( \mathbb{E}_t[\pi^p(t + \Delta t, y(t + \Delta t))] + \frac{1}{2\gamma} \mathbb{V}\text{ar}_t[\pi^p(t + \Delta t, y(t + \Delta t))] \right). \tag{2.18}
$$

Hence, we assume that we want to measure the variance of $\pi^p()$ relative to the expected value of $\pi^p()$. Obviously, this is only well-defined if $\pi^p(t, y)$ is strictly positive for all $(t, y)$.

If we employ our Taylor-expansion and take the limit for $\Delta t \downarrow 0$ we obtain the following pde

$$
\pi^p_t + a\pi^p_y + \frac{1}{2}b^2\pi^p_{yy} + \frac{1}{2}q^{-1}v^p(\pi^p_y)^2 - r\pi^p = 0. \tag{2.19}
$$

Again, we can study the solution of (2.19) by employing a transformation of the solution that removes the non-linearity from the pde. Consider the auxiliary function $h^p(t, y) := (\pi^p(t, y))^{1/q}$. The original function can be obtained from the inverse relation $\pi^p(t, y) = (h^p(t, y))^q$. If we now apply the chain-rule of differentiation, we can express the partial derivatives of $\pi^p()$ in terms of $h^p()$ as

$$
\pi^p_t = q(h^p)^{q-1}h^p_t, \quad \pi^p_y = q(h^p)^{q-1}h^p_y, \\
\pi^p_{yy} = q(h^p)^{q-1} \left( \frac{q - 1}{h^p} (h^p_y)^2 + h^p_{yy} \right). \tag{2.20}
$$

If we substitute these expressions into (2.19) and simplify, we obtain

$$
h^p_t + ah^p_y + \frac{1}{2}b^2 \left( \frac{q - 1 + \gamma q}{h^p} (h^p_y)^2 + h^p_{yy} \right) - \frac{r}{q}h^p = 0. \tag{2.21}
$$

If we choose $q = 1/(1 + \gamma)$, then the non-linear terms cancel and we obtain a linear pde for $h^p(t, y)$:

$$
h^p_t + ah^p_y + \frac{1}{2}b^2h^p_{yy} - r(1 + \gamma)h^p = 0. \tag{2.22}
$$
The boundary condition at $T$ is given by $h^p(T, y(T)) = \pi^p(T, y(T))^{1+\gamma} = f(y(T))^{1+\gamma}$. Using the Feynman-Kac formula, we can express the solution of (2.22) as

$$h^p(t, y) = \mathbb{E}_t \left[ e^{-r(1+\gamma)(T-t)} f(y(T))^{1+\gamma} | y(t) = y \right], \quad (2.23)$$

where the expectation is taken with respect to the stochastic process $y(t)$ defined in equation (2.1) conditional on the information that at time $t$ the process $y(t)$ is equal to $y$. From the representation (2.23) follows immediately that we can express $\pi^p(t, y)$ as

$$\pi^p(t, y) = e^{-r(T-t)} \left( \mathbb{E}_t \left[ f(y(T))^{1+\gamma} | y(t) = y \right] \right)^{\frac{1}{1+\gamma}}, \quad (2.24)$$

Note that this representation of the price $\pi^p()$ arises also in the study of indifference pricing under power-utility functions, and the related notion of pricing under so-called “$q$-optimal” measures. See, for example Hobson (2004) and ?.

### 2.2.4 Mean Value Principle

The examples we gave in the previous subsections, are all special cases of the **Mean Value Principle**, which is defined as

$$\Pi^m_t[f(y(T))] = v^{-1} \left( \mathbb{E}_t[v(f(y(T)))] \right) \quad (2.25)$$

for any function $v()$ which is a convex and increasing (see Kaas et al., 2008, Chap. 5).

Once more, we have to pay attention to units. If we want to apply a general function $v()$ to a value (expressed in units of \$), we have to make sure that the argument of $v()$ is dimensionless. The easiest way to achieve this, is to express the argument for $v()$ in “forward terms”. For a single binomial step, we therefore obtain the following expression for the price:

$$\frac{\pi^m(t, y(t))}{e^{rt}} = v^{-1} \left( \mathbb{E}_t \left[ v \left( \frac{\pi^m(t + \Delta t, y(t + \Delta t))}{e^{r(t+\Delta t)}} \right) \right] \right). \quad (2.26)$$

We can rewrite this definition as

$$v \left( \frac{\pi^m(t, y(t))}{e^{rt}} \right) = \mathbb{E}_t \left[ v \left( \frac{\pi^m(t + \Delta t, y(t + \Delta t))}{e^{r(t+\Delta t)}} \right) \right], \quad (2.27)$$

and from this expression it is immediately obvious that the “distorted” value $v(\pi^m(t, y)/e^{rt})$ is linear and therefore satisfies the Feynman-Kac formula. Which is exactly the solutions we found in the previous subsections.
In this case, we want to go in the opposite direction and we seek the corresponding pde for the price $\pi^m(t, y)$. To do this, we will use our Taylor-expansion derivation, but with an additional twist: we must expand the functions $v()$ and $v^{-1}()$ as well. In particular, we seek to expand the function $v^{-1}(v(x) + h)$ for small $h$. Using the identity $v^{-1}(v(x)) \equiv x$ we obtain

$$v^{-1}(v(x) + h) = x + \frac{h}{v'(x)} - \frac{v''(x)}{2(v'(x))^3} h^2 + O(h^3). \quad (2.28)$$

Combining this result with the Taylor expansions for $v()$ and $\pi^m()$ on a single binomial time-step, and taking the limit for $\Delta t \downarrow 0$ leads to the pde:

$$\pi_t^{mf} + a\pi_y^{mf} + \frac{1}{2} b^2\pi_{yy}^{mf} + \frac{1}{2} \frac{v''(\pi^{mf})}{v'(\pi^{mf})} (b\pi_y^{mf})^2 = 0, \quad (2.29)$$

where $\pi^{mf}(t, y) := \pi^m(t, y)/e^{rt}$ is the price expressed in forward terms.

We see in equation (2.29), that the coefficient in front of the non-linear term can be identified as the “local risk aversion” induced by the function $v()$ at the current value $\pi^{mf}()$. Note, that since the function $v()$ is increasing and convex by assumption, $v''()/v'()$ is positive.

### 2.2.5 BSDE’s and g-expectations

The non-linear pde’s we have derived in (2.9), (2.16), (2.19) and (2.29) (and will derive in the sections to come) have been intensively studied in recent years in the context of backward stochastic differential equations (or BSDE’s). Necessary conditions for the existence and uniqueness of solutions of BSDE’s have been established. For an overview of applications of BSDE’s in finance, we refer to El Karoui et al. (1997).

Using “BSDE notation” one can show that the solution to the pde (2.16) can be represented by the triplet of processes $(y_t, Y_t, Z_t)$ satisfying

$$
\begin{cases}
    dy_t = a(t, y_t) \, dt + b(t, y_t) \, dW_t \\
    dY_t = -g(t, y_t, Y_t, Z_t) \, dt + Z_t \, dW_t \\
    Y_T = f(y(T))
\end{cases},
$$

with “generator” $g(t, y, Y, Z) = \frac{1}{2} b^2 \pi_{yy}^{mf} Z^2 - rY$. The realisation of the stochastic process $Y_t$ (depending on $y_t$) is then considered to be the solution to (2.16). Another example: to represent the solution to (2.29) of the price under the mean value principle for a given function $v()$, we would use a BSDE with generator $g(t, y, Y, Z) = \frac{1}{2} (v''(Y)/v'(Y)) Z^2$.

The mathematical setup of BSDE’s allows significant generalisations over the restrictive Markovian diffusion setting we are using in this paper.
For example, one can extend the setup to Lévy processes, see Nualart and Schoutens (2001). However, we will not explore that avenue in this paper.

Furthermore, solutions to BSDE’s are always time-consistent, and can therefore be used to define non-linear “g-expectations” (where the prefix “g” refers to the generator $g(t, y, Y, Z)$ in (2.30)) and the related notion of “g-martingales”, see, for example, Peng (2004). The connection between $g$-expectations and time-consistent risk measures has been explored in a recent paper by Rosazza Gianin (2006).

** Note: StDev price is apparently special case of $g$-expectation, known as a “$\mu\mid Z$-expectation” with $g(t, y, Y, Z) = \mu\mid Z$ and notation $E^\mu[]$. Consequently, buyer’s and seller’s price corresponds to $E^\mu[]$ and $E^{\mu\mid \cdot}[]$. (Coquet paper)

### 2.3 Standard-Deviation Principle

Another well-known actuarial pricing principle is the Standard Deviation Principle, (see Kaas et al., 2008) defined as

$$
\Pi^s_t[f(y(T))] = E_t[f(y(T))] + \beta \sqrt{\text{Var}_t[f(y(T))]}.
$$

(2.31)

Please note that also in this case we have to be careful about the dimensionality of the parameter $\beta$. Even though both the expectation and the standard deviation are expressed in units of $\mathcal{E}$, the standard deviation and the expectation have different “time-scales”. If we go down to small time-scales (as we will be doing when considering the limit for $\Delta t \downarrow 0$) then due to the diffusion term $dW$ of the process $y$, we have the property that the expectation of any function $f(y)$ scales linearly with $\Delta t$, but the standard deviation scales with $\sqrt{\Delta t}$. This means that for small $\Delta t$ the standard deviation term will completely overpower (literally!) the expectation term. Therefore, the only way to obtain a well-defined limit for $\Delta t \downarrow 0$ is if we take $\beta \sqrt{\Delta t}$ as the parameter for the standard deviation principle for a binomial step.

Another way of understanding this result, is to consider the following example. If we want to compare a standard deviation measured over an annual time-step with a standard deviation measured over a monthly time-step, we have to scale the annual outcome with $\sqrt{1/12}$ to get a fair comparison.

Given the above discussion on the time scales, we get for a single binomial step, the following expression for the price:

$$
\pi^s(t, y(t)) = e^{-r\Delta t} \left( E_t[\pi^s(t + \Delta t, y(t + \Delta t))] + \beta \sqrt{\Delta t} \sqrt{\text{Var}_t[\pi^s(t + \Delta t, y(t + \Delta t))]} \right).
$$

(2.32)
Using a similar derivation as in Section 2.2, we arrive at the following partial differential equation for $\pi^s(t, y)$:

$$\pi^s_t + a\pi^s_y + \frac{1}{2} b^2 \pi^s_{yy} + \beta \sqrt{(b\pi^s_y)^2} - r\pi^s = 0,$$

(2.33)

which can be rewritten as

$$\pi^s_t + a\pi^s_y + \frac{1}{2} b^2 \pi^s_{yy} + \beta |\pi^s_y| - r\pi^s = 0.$$

(2.34)

This is again once more a non-linear pde. However, the non-linearity is much more benign in this case. Whenever the partial derivative $\pi^s_y$ does not change sign on the whole domain of $y$ (i.e. the function $\pi^s$ is either monotonically increasing or monotonically decreasing in $y$), then (2.34) reduces to the linear pde:

$$\pi^s_t + (a \pm \beta b)\pi^s_y + \frac{1}{2} b^2 \pi^s_{yy} - r\pi^s = 0,$$

(2.35)

where the sign of $\pm \beta b$ depends on the (uniquely defined) sign of $\pi^s_y$.

Using the Feynman-Kac formula, we can represent the solution to (2.35) as:

$$\pi^s(t, y) = \mathbb{E}^{S\tau}_{t}[f(y(T))|y(t) = y],$$

(2.36)

where $\mathbb{E}^{S\tau}$ denotes the expectation at time $t$ with respect to the “risk-adjusted” process $y$ defined as

$$dy = \left(a(t, y) \pm \beta b(t, y)\right) dt + b(t, y) dW^S,$$

(2.37)

The drift-rate is adjusted upwards $(a + \beta b)$ if the payoff $f(y)$ is monotonically increasing in $y$, and adjusted downwards $(a - \beta b)$ if $f(y)$ is monotonically decreasing in $y$. So, the risk-adjustment is always in the “upwind” direction of the risk, thus making the price $\pi^s$ more expensive than the real-world expectation $\mathbb{E}[f(y)]$.

### 2.4 Cost-of-Capital Principle

Another actuarial pricing principle is the Cost-of-Capital Principle. This was introduced by the Swiss insurance supervisor as a part of the method to calculate solvency capitals for insurance companies (Keller and Luder, 2004). The Cost-of-Capital method has been widely adopted by the insurance industry in Europe, and has also been prescribed as the standard method by the European Insurance and Pensions Supervisor for the Quantitative Impact Studies (see CEIOPS, 2008).

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2For a critical discussion on the risk-measure implied by the Swiss Solvency Test we refer to Filipovic and Vogelpoth (2008).
The Cost-of-Capital is based on the following economic reasoning. We first consider the “expected loss” $E[f(y(T))]$ of the insurance claim $f(y(T))$ as a basis for pricing. But this is not enough, the insurance company also has to hold a capital buffer against the “unexpected loss”. This buffer is calculated as a Value-at-Risk over a time-horizon (typically 1 year) and a probability threshold $q$ (usually 0.995, or even higher). The unexpected loss is then calculated as $\text{VaR}_q[f(y(T)) - E[f(y(T)]]$. The capital buffer is borrowed from the shareholders of the insurance company (i.e. the buffer is subtracted from the surplus in the balance sheet). Given the very high confidence level, in most cases the buffer can be returned to the shareholders, however there is a small probability $(1 - q)$ that the capital buffer is needed to cover an unexpected loss. Hence, the shareholders require a compensation for this risk in the form of a “cost-of-capital”. This cost-of-capital needs to be included in the pricing of the insurance contract. If we denote the cost-of-capital by $\delta$, then the Cost-of-Capital Principle is given by

$$\Pi_t[f(y(T))] = E_t[f(y(T))] + \delta \text{VaR}_{q,t}[f(y(T)) - E_t[f(y(T)]]]. \quad (2.38)$$

Please note that also in this case we have to be careful about the dimensionality of the different terms. First, we are comparing Value-at-Risk quantities at different time-scales, and these have to be scaled back to a per annum basis, to do this we divide the VaR-term by $\sqrt{\Delta t}$. Then, we must realise that the cost-of-capital $\delta$ behaves like an interest rate: it is the compensation the insurance company needs to pay to its shareholders for borrowing the buffer capital over a certain period. The cost-of-capital is expressed as a percentage per annum, hence over a time-step $\Delta t$ the insurance company has to pay a compensation of $\delta \Delta t$ per $€$ of buffer capital. As a result, we obtain a “net scaling” of $\delta \Delta t/\sqrt{\Delta t} = \delta \sqrt{\Delta t}$. Note, that this is the same scaling as for the standard deviation principle.

For a single time-step, we therefore get the following expression for the cost-of-capital price:

$$\pi^c(t, y(t)) = e^{-r \Delta t} \left( E_t[\pi^c(t + \Delta t, y(t + \Delta t))] + \delta \sqrt{\Delta t} \text{VaR}_{q,t}[\pi^c(t + \Delta t, y(t + \Delta t)) - E_t[\pi^c(t + \Delta t, y(t + \Delta t))]] \right). \quad (2.39)$$

In the previous sections we used a binomial discretisation of the process $y$. However, in this case we have to be a bit more careful. Since we are considering a $(1 - q)$-quantile with very small probability $(1 - q)$, a simple binomial tree approximation is too crude to obtain an accurate representation of the $(1 - q)$-quantile of the process $y$. We therefore consider a “quadrinomial” tree, where
we make sure that we match the mean, the variance and the $(1-q)$-quantile of the process $y$ over a $\Delta t$ time-step. We can do this, for example, by considering the following discretisation:

$$y(t + \Delta t) = y(t) + a\Delta t + \begin{cases} +kb\sqrt{\Delta t} & \text{with prob. } (1-q) \\ +lb\sqrt{\Delta t} & \text{with prob. } \frac{1}{2} - (1-q) \\ -lb\sqrt{\Delta t} & \text{with prob. } \frac{1}{2} - (1-q) \\ -kb\sqrt{\Delta t} & \text{with prob. } (1-q) \end{cases}$$

(2.40)

where $l = \sqrt{\frac{1}{2} - (1-q)k^2}$.

and where $k$ is chosen such that $kb\sqrt{\Delta t}$ matches the $(1-q)$-quantile of the random variable $y(t + \Delta t) - \mathbb{E}_t[y(t + \Delta t)]$ for the time-step $\Delta t$. In particular, the distribution for $y(t + \Delta t)$ will be very close to a Gaussian for small $\Delta t$, and we can compute $k$ from the Gaussian distribution function $\Phi()$ as $k = \Phi^{-1}(q)$. For example, if $q = 0.995$, then $k = 2.58$ and $l = 0.971$ (rounded to three significant digits).

Given the quadrinomial discretisation (2.40), we can now proceed as in the previous sections, and derive a pde for the price operator $\pi^c(t,y)$:

$$\pi_c^t + a\pi_c^y + \frac{1}{2}b^2\pi_{yy} + \delta k b|\pi_c|^y - r\pi_c = 0.$$  

(2.41)

This pde is exactly the same as (2.34), except for the factor $\delta k$ instead of $\beta$ in front of $b|\pi_c|^y$. Of course, this should not come as a surprise, since for a small time-step $\Delta t$ the $(1-q)$-quantile of $y(t + \Delta t)$ converges to $k$ times the standard deviation $b\sqrt{\Delta t}$, and hence the cost-of-capital pricing operator $\pi^c$ should converge to the standard deviation pricing operator $\pi^s$ with $\beta = \delta k$.

If the payoff $f(y(T))$ is monotonous in $y(T)$, we can represent the cost-of-capital price $\pi^c(t,y)$ in the same way as the standard-deviation price (2.36) with respect to the “risk-adjusted” process $y$

$$dy = (a(t,y) \pm \delta kb(t,y)) dt + b(t,y) dW.$$  

(2.42)

### 2.5 Davis Price

So far, we have established that the continuous-time limit of the iterated variance price $\pi^v()$ solves the non-linear pricing pde (2.16). The continuous-time limit of the standard deviation principle $\pi^s()$ (and also the Cost-of-Capital principle) solves the linear pricing pde (2.35), which is considerably easier to solve. In this section we provide a connection between these two pricing principles. We show that the linear standard deviation price can be interpreted
as the “small perturbation” expansion of non-linear variance price. The core of this idea can be traced back to Davis (1997).

Let us consider the small perturbation expansion in more detail. Suppose we already have an existing portfolio of insurance liabilities, where the variance price $\pi_v(t, y)$ has already been determined for all relevant $t$ and $y$. Consider now a small position in an additional insurance claim with payoff $\varepsilon g(y(T))$ at time $T$. Let us assume that for small $\varepsilon$ the total price of the insurance portfolio can be decomposed into $\pi_v(t, y) + \varepsilon \pi^D(t, y)$, where $\pi^D()$ denotes the “Davis-price” of the additional claim $\varepsilon g()$. The total price $\pi_v() + \varepsilon \pi^D()$ should solve the pricing pde (2.16), and we find

\[
(\pi^v_t + \varepsilon \pi^D_t) + a(\pi^v_y + \varepsilon \pi^D_y) + \frac{1}{2} b^2(\pi^v_{yy} + \varepsilon \pi^D_{yy}) + \frac{1}{2} \frac{\gamma}{X_0 e^{rt}} b^2 \left((\pi^v_y)^2 + 2 \varepsilon \pi^v_y \pi^D_y + \varepsilon^2 (\pi^D_y)^2\right) - r (\pi^v_v + \varepsilon \pi^D_v) = 0. \tag{2.43}
\]

By definition, the price $\pi^v()$ solves the pde, and we simplify the expression to

\[
\pi^D_t + a\pi^D_y + \frac{1}{2} b^2 \pi^D_{yy} + \frac{1}{2} \frac{\gamma}{X_0 e^{rt}} b^2 \left(2 \pi^v_y \pi^D_y + \varepsilon (\pi^D_y)^2\right) - r\pi^D = 0. \tag{2.44}
\]

For small $\varepsilon$, we can ignore the non-linear $\varepsilon$-term, and we obtain a linear pde for the small perturbation Davis-price

\[
\pi^D_t + \left(a + \frac{\gamma}{X_0 e^{rt}} b^2 \pi^v_y\right) \pi^D_y + \frac{1}{2} b^2 \pi^D_{yy} + r \pi^D = 0. \tag{2.45}
\]

Using the Feynman-Kač formula, we can represent the solution to (2.45) as:

\[
\pi^D(t, y) = E^D_t [f(y(T))|y(t) = y], \tag{2.46}
\]

where $E^D_t[]$ denotes the expectation at time $t$ with respect to the “risk-adjusted” process $y$ defined as

\[
dy = \left(a(t, y) + \frac{\gamma}{X_0 e^{rt}} b^2(t, y) \pi^v_y(t, y)\right) dt + b(t, y) dW^D. \tag{2.47}
\]

Note, that the Davis-price defined only “relative” to existing portfolio price $\pi^v()$. In particular, we can interpret the adjustment to the drift term as the risk-aversion $\gamma/X_0 e^{rt}$ times the local “standard deviation” of the existing price process $b(t, y) \pi^v_y(t, y)$ times the “standard deviation” $b(t, y)$ of the insurance process $y$ the drives the additional claim $g(y(T))$. 
3 Combining Financial and Insurance Risk

In this section, we want to push our analysis one step further. We seek
to investigate what happens to the pricing formulæ when we consider an
environment where we have both financial risk that can traded and hedged
in a market, and unhedgeable insurance risk.

To keep our expositions as simple as possible, we model the financial risk
in a Black and Scholes (1973) economy. We model the stock price $S_t$ via its
return process $x_t = \ln S_t$, which is given by a diffusion equation:

$$dx = \left(\mu(t, x) - \frac{1}{2}\sigma^2(t, x)\right)dt + \sigma(t, x) \, dW_f,$$  \tag{3.1}

where the Brownian Motion $W_f$ is a new Brownian Motion that is possibly
correlated to the Brownian Motion $W$ that drives the insurance market.

The Black and Scholes (1973) economy is arbitrage-free and complete.
This means that there exists a unique equivalent probability measure $Q$
that defines the (linear) no-arbitrage pricing operator

$$\pi^Q(t, x) := e^{-r(T-t)}E^Q_t[F(x(T))]$$  \tag{3.2}

for a (financial) derivative with payoff $F(x(T))$ at time $T$. Under the mea-
sure $Q$ the return process $x_t$ is given by

$$dx = \left(r - \frac{1}{2}\sigma^2(t, x)\right)dt + \sigma(t, x) \, dW^Q_f,$$  \tag{3.3}

where $r$ is the risk-free interest rate.

We also consider a discretisation scheme for the return process in the form
of a binomial tree:

$$x(t + \Delta t) = x(t) + \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \begin{cases}
+\sigma\sqrt{\Delta t} & \text{with } P\text{-prob. } \frac{1}{2} \\
-\sigma\sqrt{\Delta t} & \text{with } P\text{-prob. } \frac{1}{2}
\end{cases}$$  \tag{3.4}

where we have (once more) suppressed the dependence on $(t, x)$. This dis-
cretisation is valid under the real-world probability measure $P$. We also wish
to consider the process for $x$ under the equivalent measure $Q$. Under this
measure the mean of the process for a time-step $\Delta t$ should be $(r - \frac{1}{2}\sigma^2)\Delta t$.
Therefore, under the measure $Q$ the process $x$ should be discretised as

$$x(t + \Delta t) = x(t) + \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \begin{cases}
+\sigma\sqrt{\Delta t} & \text{with } Q\text{-prob. } \frac{1}{2} \left(1 - \frac{\mu - r}{\sigma}\sqrt{\Delta t}\right) \\
-\sigma\sqrt{\Delta t} & \text{with } Q\text{-prob. } \frac{1}{2} \left(1 + \frac{\mu - r}{\sigma}\sqrt{\Delta t}\right)
\end{cases}$$  \tag{3.5}
The quantity \((\mu - r)/\sigma\) that drives the change in probabilities from \(P\) to \(Q\) has the economic interpretation as the *market-price of financial risk*.

The two processes \(x\) and \(y\) can be jointly discretised in a “quadrinomial” tree (under the real-world measure \(P\)) as illustrated in Figure 1. Using these probabilities we make sure that the correlation between the processes \(x\) and \(y\) is equal to \(\rho\), and that the marginal probabilities for each individual process are equal to the binomial \(P\)-probabilities of \(\frac{1}{2}\).

### 3.1 Market-Consistent Pricing

Given our definitions of the hedgeable financial process \(x\) and the unhedgeable insurance process \(y\), we want to turn our attention to market-consistent pricing operators. Intuitively, the price \(\pi_F(t, x)\) of a pure financial claim \(F(x(T))\) should be equal to the arbitrage-free price \(e^{-r(T-t)}E^Q_t[F(x(T))]\). Furthermore, if we add a pure financial claim to a given general payoff \(G(x(T), y(T))\), then we should have that the pure financial part of the portfolio should be priced consistently with arbitrage-free pricing. We can formalise this in the following definition.

**Definition 3.1** A pricing operator \(\pi()\) is market-consistent if for any financial derivative \(F(x(T))\) and any other claim \(G(t, x, y)\) we have

\[
\pi_{F+G}(t, x, y) = e^{-r(T-t)}E^Q_t[F(x(T))] + \pi_G(t, x, y).
\]

(3.6)

Note that this definition is similar to Definition 2.16 given in Malamud et al. (2008).

Given this definition for market-consistent pricing operators, how can we now construct such operators? In general, this question has been analysed by Barrieu and El Karoui (2005, 2009). They provide characterisations in terms of inf-convolutions of risk measures. However, for the one-step variance and standard deviation valuations on the quadrinomial tree we use a simple construction from Musiela and Zariphopoulou (2009) which is based on conditional expectations.
3.2 Market-Consistent Variance Pricing

In the setting of a one-step quadrinomial tree, we can construct a market-consistent valuation as follows. Suppose we are at time $t$ and looking ahead at time $t + \Delta t$, we are given the four possible outcomes of $\pi^v(t + \Delta t) = \{\pi^v_{++}, \pi^v_{+-}, \pi^v_{-+}, \pi^v_{--}\}$. In this shorthand notation the first and second subscript denote the state of the financial and insurance process respectively.

We will do the valuation in a two-step procedure. First, we condition on the financial process $x(t + \Delta t)$, which can be in two possible states “$x^+$” and “$x^-$”. Conditional on the value $x(t + \Delta t)$, we have only pure insurance risk left, which we can price using the methods from Section 2.2.2. In the setting of the one-step quadrinomial tree we can explicitly calculate

$$\pi^v(t + \Delta t | x \pm) := \mathbb{E}[\pi^v(t + \Delta t | x \pm)] + \gamma X_0 e^{r(t + \Delta t)} \text{Var}[\pi^v(t + \Delta t | x \pm)]$$

with

$$\mathbb{E}[\pi^v(t + \Delta t | x +)] = \left(\frac{1+\rho}{2}\right) \pi^v_{++} + \left(\frac{1-\rho}{2}\right) \pi^v_{+-}$$

$$\mathbb{E}[\pi^v(t + \Delta t | x -)] = \left(\frac{1-\rho}{2}\right) \pi^v_{-+} + \left(\frac{1+\rho}{2}\right) \pi^v_{--}$$

$$\text{Var}[\pi^v(t + \Delta t | x +)] = \left(\frac{1-\rho^2}{4}\right) (\pi^v_{++} - \pi^v_{+-})^2$$

$$\text{Var}[\pi^v(t + \Delta t | x -)] = \left(\frac{1-\rho^2}{4}\right) (\pi^v_{-+} - \pi^v_{--})^2.$$ 

Note that for $\rho = 1$ or $\rho = -1$ the financial and insurance markets become perfectly correlated, and in these cases the conditional variance goes to zero. Summarising, we can say that after the first step we have used the “pure actuarial” variance principle to find the variance price (at time $t + \Delta t$) of the insurance risk conditional on the realisation of the financial risk.

In the second step, we do the arbitrage-free valuation of the financial derivative we have created in the first step:

$$\pi^v(t, x, y) = e^{-r\Delta t} \mathbb{E}^Q[\pi^v(t + \Delta t | x \pm)]$$

$$= e^{-r\Delta t} \left(\frac{1}{2} \left(1 - \frac{\mu - r}{\sigma} \sqrt{\Delta t}\right) \pi^v(t + \Delta t | x +) + \frac{1}{2} \left(1 + \frac{\mu - r}{\sigma} \sqrt{\Delta t}\right) \pi^v(t + \Delta t | x -)\right),$$

where we use the $Q$-probabilities for the discretisation of the process $x$ as defined in (3.5).

If we now employ a similar Taylor-expansion and take the limit for $\Delta t \downarrow 0$
as in Section 2, we find the following pde:

\[
\pi^v_t + (r - \frac{1}{2}\sigma^2)\pi^v_x + (a - \rho b \frac{\mu - r}{\sigma}) \pi^v_y + \\
\frac{1}{2}\sigma^2 \pi^v_{xx} + \rho \sigma b \pi^v_{xy} + \frac{1}{2}b^2 \pi^v_{yy} + \frac{\gamma}{X_0 e^{rt}} (1 - \rho^2)(b \pi^v_y)^2 - r \pi^v = 0. \tag{3.9}
\]

We clearly see in the pde the impact of the market-consistency, because the process has the arbitrage-free drift-term \((r - \frac{1}{2}\sigma^2)\) from the measure \(Q\). Note that the insurance process is evaluated using a new drift term \((a - \rho b (\mu - r)/\sigma)\), which is consistent with the change of measure to \(Q\) for the “financial” Brownian Motion \(W_f\). Furthermore, a non-linear term is added that reflects the “local unhedgeable variance” \((1 - \rho^2)(b \pi^v_y)^2\) of the insurance risk.

Unfortunately, we cannot find a general solution for the non-linear pde (3.9), (other than the representations in terms of BSDE’s discussed in Section 2.2.5). However, we can find explicit solutions for some interesting special cases.

### 3.2.1 Special Case: Pure Insurance Payoff

The first special case is the case considered by Henderson (2002) and Musiela and Zariphopoulou (2004) for exponential indifference pricing: if we have a payoff \(f(y(T))\) that depends on insurance risk only and the market-price of financial risk \((\mu - r)/\sigma\) does not depend on \(x\), then the dependence of the pricing operator \(\pi^v(t, x, y)\) on \(x\) disappears, and we obtain a reduced pde for \(\pi^v(t, y)\):

\[
\pi^v_t + (a - \rho b \frac{\mu - r}{\sigma}) \pi^v_y + \frac{1}{2}b^2 \pi^v_{yy} + \frac{\gamma}{X_0 e^{rt}} (1 - \rho^2)(b \pi^v_y)^2 - r \pi^v = 0. \tag{3.10}
\]

Note that for \(\rho \neq 0\) the pde (3.10) is different from the “pure insurance” pde (2.16). If the insurance risk is correlated with the financial market, we can hedge a fraction of the insurance risk via the financial market, and this effect is reflected in (3.10).

The solution to (3.10) can be represented via log-transform as

\[
\pi^v(t, y) = \frac{X_0 e^{rt}}{\gamma(1 - \rho^2)} \ln \mathbb{E}^{\hat{P}} \left[ e^{\frac{\gamma(1 - \rho^2)}{X_0 e^{rt}} f(y(T))} \right] | y(t) = y, \tag{3.11}
\]

where the expectation is evaluated under a new probability measure \(\hat{P}\) where the insurance risk driver \(y\) follows the process

\[
\,dy = \left( a(t, y) - \rho b(t, y) \frac{\mu - r}{\sigma} \right) dt + b(t, y) dW^{\hat{P}}. \tag{3.12}
\]

The solution we derive here is the same as Henderson (2002) and Musiela and Zariphopoulou (2004) find for indifference pricing under exponential utility.
3.2.2 Special Case: Quadratic Payoff

Another special case of interest is whenever the payoff \( f(x(T), y(T)) \) has a quadratic structure:

\[
\pi^v(t, x, y) = A(t) + B(t)x + C(t)y + \frac{1}{2}D(t)x^2 + E(t)xy + \frac{1}{2}F(t)y^2. \tag{3.13}
\]

For such a quadratic payoff we can use the Ansatz

\[
\pi^v(t, x, y) = A(t) + B(t)x + C(t)y + \frac{1}{2}D(t)x^2 + E(t)xy + \frac{1}{2}F(t)y^2. \tag{3.14}
\]

If we substitute this Ansatz into (3.9), and collect terms of equal powers, we obtain the following system of ode’s for the coefficients \( A(t), \ldots, F(t) \):

\[
\begin{align*}
A_t - rA &+ (r - \frac{1}{2} \sigma^2)B + (a - \rho b \frac{\mu - r}{\sigma})C + \frac{1}{2} \sigma^2 D + \rho \sigma b E + \frac{1}{2} b^2 F + \frac{1}{2} X_{\omega e} \tau (1 - \rho^2) b^2 C^2 = 0 \tag{3.15a} \\
B_t - rB &+ \frac{\gamma}{X_{o e} \tau} (1 - \rho^2) b^2 C E + (r - \frac{1}{2} \sigma^2)D + (a - \rho b \frac{\mu - r}{\sigma})E = 0 \tag{3.15b} \\
C_t - rC &+ \frac{\gamma}{X_{o e} \tau} (1 - \rho^2) b^2 C F + (r - \frac{1}{2} \sigma^2)E + (a - \rho b \frac{\mu - r}{\sigma})F = 0 \tag{3.15c} \\
D_t - rD &+ \frac{\gamma}{X_{o e} \tau} (1 - \rho^2) b^2 E^2 = 0 \tag{3.15d} \\
E_t - rE &+ \frac{\gamma}{X_{o e} \tau} (1 - \rho^2) b^2 F^2 = 0 \tag{3.15e} \\
F_t - rF &+ \frac{\gamma}{X_{o e} \tau} (1 - \rho^2) b^2 F^2 = 0. \tag{3.15f}
\end{align*}
\]

This system of ode’s can be solved, provided that the coefficients \( a, b, \mu, \sigma \) are constants or functions of time only.

3.3 Market-Consistent Standard Deviation Pricing

To derive the market-consistent Standard-Deviation price (and Cost-of-Capital price), we proceed along the same line as before. Using the two-step conditioning approach, and taking the limit for \( \Delta t \downarrow 0 \) we obtain the pde

\[
\pi^s_t + (r - \frac{1}{2} \sigma^2)\pi^s_x + (a - \rho b \frac{\mu - r}{\sigma})\pi^s_y + \frac{1}{2} \sigma^2 \pi^s_{xx} + \rho \sigma b \pi^s_{xy} + \frac{1}{2} b^2 \pi^s_{yy} + \delta(1 - \rho^2)b\sqrt{\pi^s_y} - r\pi^s = 0. \tag{3.16}
\]

Under the additional assumption that \( \pi^s(t, y) \) is monotone in \( y \), we can simplify the pde to

\[
\pi^s_t + (r - \frac{1}{2} \sigma^2)\pi^s_x + (a - \rho b \frac{\mu - r}{\sigma}) \pm \delta(1 - \rho^2)b\pi^s_y + \frac{1}{2} \sigma^2 \pi^s_{xx} + \rho \sigma b \pi^s_{xy} + \frac{1}{2} b^2 \pi^s_{yy} - r\pi^s = 0. \tag{3.17}
\]

where the sign of \( \pm \delta(1 - \rho^2)b \) is given by the (uniquely defined) sign of \( \pi^s_y(t, y) \).
4 Summary and Conclusions

In this paper we have investigated well-known actuarial premium principles such as the variance principle and the standard-deviation principle, and studied their extension into both time-consistent and market-consistent directions. The method we used to construct these extensions was consider one-period valuations, then extend this to a multi-period setting using the backward iteration method of Jobert and Rogers (2008) for a given discrete time-step $\Delta t$, and finally consider the continuous-time limit for $\Delta t \to 0$. We showed that the extended variance premium principle converges to the non-linear exponential indifference valuation. Furthermore, we showed that the extended standard-deviation principle converges to an expectation under an equivalent martingale measure. Finally, we showed that the Cost-of-Capital principle, which is widely used by the insurance industry, converges to the same limit as the standard-deviation principle.
References


