Robustness, Model Uncertainty and Pricing

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Abstract

Apply ideas of robustness and model uncertainty in a context of pricing derivative contracts in complete and incomplete markets. We will focus on the (simple) case with uncertainty in mean only. First, we show that in a complete market, an agent worried about model uncertainty will choose the replicating portfolio as this will eliminate the model uncertainty completely. Hence, a perfectly rational agent that is facing model uncertainty will price risks using no-arbitrage. Second, we show that in an incomplete market the agent will hedge as much of the risk as possible and will choose a market-consistent pricing operator.

1 Introduction

Make a test reference to Carmona (2009). Also test € sign.

*The author would like to thank…
To choose the simplest possible mathematical setting, we will use a one-step binomial tree to model the uncertainty in the economy, see Figure 1. We assume that we have a perfectly rational economic agent that knows with certainty the "structural" parameters of the economy (i.e. $u$, $d$ and $R$). However, we assume that the economic agent is uncertain about the true return of the share price.

### 3.1 Tree Setup

To keep our expositions as simple as possible, we model the financial risk in a diffusion setting (similar to the Black and Scholes (1973) model). We model the stock price $S_t$ via its return process $x_t = \ln S_t$, which is given by a diffusion equation:

$$dx = m dt + \sigma dW_x,$$

where $W_x$ is a Brownian Motion. Note, that for $m = (\mu - \frac{1}{2}\sigma^2)$ we obtain the "standard" Black-Scholes parametrisation.

We wish to consider a discretisation scheme for the return process in the form of a binomial tree:

$$x(t + \Delta t) = x(t) + \left\{ \begin{array}{ll} +\sigma \sqrt{\Delta t} & \text{with prob. } \frac{1}{2}(1 + \frac{m}{\sigma} \sqrt{\Delta t}) \vspace{1ex} \\
-\sigma \sqrt{\Delta t} & \text{with prob. } \frac{1}{2}(1 - \frac{m}{\sigma} \sqrt{\Delta t}). \end{array} \right. \quad (3.2)$$

For this discretisation scheme, we can make the following identification of the
parameters of the binomial economy depicted in Figure 1:

\[
p := \frac{1}{2} (1 + \frac{m}{\sigma} \sqrt{\Delta t})
\]

\[
u := e^{\frac{m}{\sigma} \sqrt{\Delta t}}
\]

\[
d := e^{-\frac{m}{\sigma} \sqrt{\Delta t}}
\]

\[
R := e^{r \Delta t}
\]

We capture the uncertainty that the agent faces by assuming that the agent knows that \( m \in [m_L, m_H] \). This implies that the binomial probability \( p \) can be anywhere in the interval \([p_L = \frac{1}{2} (1 + \frac{m_L}{\sigma} \sqrt{\Delta t}), p_H = \frac{1}{2} (1 + \frac{m_H}{\sigma} \sqrt{\Delta t})]\). For a similar model setup, we refer to the PhD thesis by Kirch (2002).

### 3.2 Valuation with Model Uncertainty

Suppose that we have a derivative contract that generates a cash flow \( f(t + \Delta t, x(t+\Delta t)) \) at time \( t + \Delta t \). The payoff of the contract is uncertain, and depends on the realisation of the underlying process \( x(t+\Delta t) \). In the remainder of this paper we will approximate the function \( f() \) by a Taylor expansion around the point \((t, x(t))\) in the \( x \)-direction up to order \((\Delta x)^2\) (i.e. up to order \(\Delta t\)) as:

\[
f_0 = f_1 + \left\{ \begin{array}{l}
+f_x \sigma \sqrt{\Delta t} + \frac{1}{2} f_{xx} \sigma^2 \Delta t & \text{with prob. } \frac{1}{2}(1 + \frac{m}{\sigma} \sqrt{\Delta t}) \\
-f_x \sigma \sqrt{\Delta t} + \frac{1}{2} f_{xx} \sigma^2 \Delta t & \text{with prob. } \frac{1}{2}(1 - \frac{m}{\sigma} \sqrt{\Delta t}),
\end{array} \right.
\]

with the shorthand notations \( f_0 := f(t, x(t)), f_1 := f(t+\Delta t, x(t)), f_x := \partial f(t, x(t))/\partial x \) and \( f_{xx} := \partial^2 f(t, x(t))/\partial x^2 \).

What value should the agent attribute to this contract at time 0? If there would be no uncertainty about the drift \( m \) of the process \( x \), the rational agent would first determine the certainty equivalent \( \mathbb{E}_t[f(t+\Delta t, x(t+\Delta t))] = f_1 + (f_x m + \frac{1}{2} f_{xx} \sigma^2) \Delta t \) of the uncertain payoff at time \( t + \Delta t \), and then determine the value at time \( t \) as the discounted certainty equivalent \( e^{-r \Delta t} (f_1 + (f_x m + \frac{1}{2} f_{xx} \sigma^2) \Delta t) \). [References?]

However, given the uncertainty about the correct drift \( m \), a robust approach for the rational agent would be to consider the “worst case” discounted certainty equivalent \( \min_{m \in [m_L, m_H]} e^{-r \Delta t} \mathbb{E}_t[f(t+\Delta t, x(t+\Delta t))] \). In the very simple case of the binomial tree, the solution of the minimisation problem can be expressed explicitly as

\[
\left\{ \begin{array}{l}
e^{-r \Delta t} (f_1 + (f_x m_L + \frac{1}{2} f_{xx} \sigma^2) \Delta t) & \text{if } f_x > 0 \\
e^{-r \Delta t} (f_1 + (\frac{1}{2} f_{xx} \sigma^2) \Delta t) & \text{if } f_x = 0 \\
e^{-r \Delta t} (f_1 + (f_x m_H + \frac{1}{2} f_{xx} \sigma^2) \Delta t) & \text{if } f_x < 0.
\end{array} \right.
\]
Note that a robust rational agent acts as if a malevolent “mother nature” chooses the worst possible outcome for any given the payoff \( f() \), and therefore the “optimal” drift term is either \( m_L \) or \( m_H \) depending on the sign of \( f_x \).

Note, that this method for pricing is well-known by actuaries. Using the “pessimistic” outcome for the uncertain drift \( m \) is known to actuaries as prudence.

3.3 Hedging

Suppose now that the rational agent can trade in the share price \( S \). This means that the agent buys an initial position \( \theta \) in the share (i.e. the agent buys \( \theta / S(t) \) shares at \( t \)), which is financed by borrowing an amount \( \theta \) from the bank account \( B \). At time \( t + \Delta t \), the net position in shares (after repaying the loan plus interest \( e^{r \Delta t} \)) has a value of \((e^{x(t+\Delta t)}-f(t)-e^{r \Delta t} \theta)\).

The agent now wishes to consider what is the optimal amount \( \theta \) to invest in shares that maximises the net payoff. The optimisation problem therefore can be stated as (ignoring terms of higher order than \( \Delta t \)):

\[
\max_{\theta} \min_{m \in [m_L, m_H]} e^{-r \Delta t}(f_1 + (f_x m + \frac{1}{2} f_{xx} \sigma^2 + (m + \frac{1}{2} \sigma^2 - r) \theta) \Delta t). \tag{3.5}
\]

The optimal policy pair \((\theta, m)\) for this optimisation problem can be found by considering the partial derivatives with respect to \( \theta \) and \( m \), which are given by

\[
\frac{\partial}{\partial \theta} : e^{-r \Delta t}(m + \frac{1}{2} \sigma^2 - r) \Delta t, \tag{3.6}
\]

\[
\frac{\partial}{\partial m} : e^{-r \Delta t}(f_x + \theta) \sigma \Delta t. \tag{3.7}
\]

**Necessary condition needed:**

\[
m_L < r - \frac{1}{2} \sigma^2 \tag{3.8a}
\]

\[
m_H > r - \frac{1}{2} \sigma^2 \tag{3.8b}
\]

The optimal choice for \( m \) of the inner minimisation problem depends on the sign of \( (3.7) \). In particular, if \( (3.7) \) is positive, then the value is minimised by setting \( m \) as low as possible at \( m_L \) and vice-versa. However, the sign of \( (3.7) \) is determined by \( \theta \). In particular, equation \( (3.7) \) is zero for the “magical” value \( \theta = -f_x \), and negative for \( \theta < -f_x \) (and positive for \( \theta > -f_x \)).

Suppose the agent considers a value \( \theta > -f_x \). In this case the sign of \( (3.7) \) is positive, and mother nature’s optimal choice for \( m \) is \( m = m_L \). As a result, the agent should consider the sign of \( (3.6) \), which is now given by \( e^{-r \Delta t}(m_L \sigma + \frac{1}{2} \sigma^2 -

\footnote{Optimal hedging problems of this type have also been considered by Barrieu and El Karoui (2005, 2009).}
and is negative by assumption (3.8a). Hence, the agent can increase the value of her position by decreasing $\theta$ to $\theta = -f_x$.

On the other hand, suppose the agent considers a value $\theta < -f_x$. In this case the sign of (3.7) is negative, and mother nature chooses $m = m_H$. As a result, the sign of (3.6) is positive by assumption (3.8b). Hence, the agent can increase the value of her position by increasing $\theta$ to $\theta = f_x$.

We must therefore conclude that (given assumption (3.8)) the optimal choice for the rational agent is to go short an amount $f_x$ in shares. But this is a very interesting conclusion, as this is the delta-hedging strategy that perfectly replicates the derivative contract $f()$. In fact, exactly for this choice, the agent’s exposure to the malevolent actions of mother nature is eliminated, and the price the rational agent will charge is the “arbitrage-free” price $e^{-rt}(f_1 + (r - \frac{1}{2}\sigma^2)f_x + \frac{1}{2}\sigma^2f_{xx}\Delta t)$.

Discussion on assumption (3.8). What if violated?

In a sense we can therefore conclude that (in a complete market setting) robustness and model uncertainty automatically give rise to arbitrage-free valuation.

4 Trading Restrictions: Incomplete Market

In the previous section we made the assumption that the agent can buy and sell unlimited amounts of the financial instrument $S$ at the same price. Although mathematically convenient, this is only a (crude) approximation of reality. In this section we want to introduce frictions in the financial market, that cause the financial market to be incomplete.

4.1 Constraint on Hedging Instrument

Introduce upper and lower bounds on trading in financial instrument.

$$\max_{\theta \in [\theta_L, \theta_H]} \min_{m \in [m_L, m_H]} e^{-rt}(f_1 + (f_xm + \frac{1}{2}f_{xx}\sigma^2 + (m + \frac{1}{2}\sigma^2 - r)\theta)\Delta t).$$  \hspace{1cm} (4.1)

The first order conditions are still given by (3.7) and (3.6), and equation (3.7) will be zero for the “magical” value $\theta = -f_x$. For $-f_x \in [\theta_L, \theta_H]$, the trading constraints are not binding, and we arrive at exactly the same conclusions as in Section 3.3.

Suppose now that $-f_x > \theta_H$. In this case the agent can do no better than to set $\theta = \theta_H$, and mother nature will choose $m = m_H$ (since $\theta_H < -f_x$ makes the sign of (3.7) negative). We can represent the optimised value for (4.1) as

$$e^{-rt}(f_1 + (r - \frac{1}{2}\sigma^2)\theta_H + m_H(f_x - \theta_H) + \frac{1}{2}\sigma^2f_{xx}\Delta t).$$  \hspace{1cm} (4.2)
In this expression the term \((r - \frac{1}{2}\sigma^2)\theta_H\) represents the “hedgeable” part of the risk, that is priced by replication in a market-consistent way. On the other hand, the term \(m_H(f_x - \theta_H)\) prices the unhedgeable excess position \((f_x - \theta_H)\) using the “worst case” drift \(m_H\) (since \(f_x - \theta_H < 0\)).

Obviously, the reverse conclusion holds for the case \(-f_x < \theta_L\), and we obtain the pricing formula

\[
e^{-r\Delta t}(f_1 + ((r - \frac{1}{2}\sigma^2)\theta_L + m_L(f_x - \theta_L) + \frac{1}{2}\sigma^2 f_{xx})(\Delta t)).
\] (4.3)

Interpret this in terms of “projection result” of risk measures, see Barrieu and El Karoui (2009, p. 92 and p. 140) for a discussion. Also note, that we now switch from coherent to convex risk measure.

4.2 Bid-Ask Spreads

Still to do.

Idea: introduce two assets: a “long” asset \(S_l\) and a “short” asset \(S_s\) driven by the return processes \(x_l(t + \Delta t) = x(t + \Delta t) - \alpha \Delta t\) and \(x_s(t + \Delta t) = x(t + \Delta t) + \alpha \Delta t\), where \(\alpha \Delta t\) denotes the proportional transaction costs. Furthermore we have trading restrictions \(\theta_l \in [0, \theta_H]\) and \(\theta_s \in [-\theta_L, 0]\).

Analysis should carry through in the same fashion as before, but a bit more messy...

Unrestricted case \(\theta_l \in [0, \infty]\) and \(\theta_s \in [-\infty, 0]\) should give the same pricing as with different rate for borrowing and lending. See also paper by El Karoui et al. (1997) and Cvitanić and Karatzas (1993).

5 One-Step Quadrinomial Tree: Incomplete Market

In this section, we want to push our analysis one step further. We seek to investigate how the robust rational agent acts when we consider an environment where we have both financial risk that can traded and hedged in a market (as in the previous section), and additionally an unhedgeable insurance risk.

5.1 Tree Setup

To keep our expositions as simple as possible, we model the financial and insurance risk in a diffusion setting (similar to the Black and Scholes (1973) model). As in Section 3.1, we model the stock price \(S_t\) via its return process \(x_t = \ln S_t\), which is given by the diffusion equation (3.1).

We also wish to consider an insurance process given by a diffusion equation:

\[
dy = adt + bdW_y,
\] (5.1)
Figure 2: Quadrinomial probabilities for joint processes \( x \) and \( y \).

where the Brownian Motion \( W_y \) is correlated to \( W_x \) by \( dW_x \, dW_y = \rho \, dt \).

We also consider a discretisation scheme for the insurance process in the form of a binomial tree:

\[
y(t + \Delta t) = y(t) + \begin{cases} 
  +b\sqrt{\Delta t} & \text{with prob. } \frac{1}{2}(1 + \rho \sqrt{\Delta t}) \\
  -b\sqrt{\Delta t} & \text{with prob. } \frac{1}{2}(1 - \rho \sqrt{\Delta t})
\end{cases}
\]  

(5.2)

The two processes \( x \) and \( y \) can be jointly discretised in a “quadrinomial” tree with probabilities \((p_{++}, p_{+-}, p_{-+}, p_{--})\) as illustrated in Figure 2. Using these probabilities we ensure that the correlation between the processes \( x \) and \( y \) is equal to \( \rho \), and that the marginal probabilities for each individual process are equal to the binomial probabilities of given in (3.2) and (5.2) respectively.

Note that for all probabilities we have derived so far, it can be the case that we obtain negative outcomes for some parameter values \( m \) and \( a \). However, we will silently assume that \( \Delta t \) is small enough to make all probabilities positive.

### 5.2 Model Uncertainty and Hedging

Similar to the setup in Section 3, we want to consider a rational agent that is facing model uncertainty. The agent is uncertain about the true value of the drift parameters \( m \) and \( a \) of the financial and the insurance processes. We assume that the agent faces no uncertainty about the diffusion coefficients \( \sigma \), \( b \) and the correlation parameter \( \rho \).

To help us describe the uncertainty set, we introduce some further notation. We define the vector of drift rates \( \mu \), and the covariance matrix \( \Sigma \) as follows

\[
\mu := \begin{pmatrix} m \\ a \end{pmatrix}, \quad \Sigma := \begin{pmatrix} \sigma^2 & \rho \sigma b \\ \rho \sigma b & b^2 \end{pmatrix}.
\]  

(5.3)

We now make the assumption, that the uncertainty in the drift rates is described by the following set

\[
\mathcal{K} := \{ \mu_0 + \varepsilon \mid \varepsilon' \Sigma^{-1} \varepsilon \leq k^2 \}.
\]  

(5.4)
The specification of the uncertainty in this form is motivated by the fact that the economic agent can use econometric estimation techniques to estimate the drift rates. The estimation leads to the point-estimate $\mu_0$. However, there is uncertainty surrounding this estimate. This uncertainty typically is proportional to the covariance matrix $\Sigma$. In other words, the agent assumes that the “true” value of the drift parameters lie somewhere within the confidence interval given by the set $\mathcal{K}$. In the 1-dimensional case we considered in Section 3, the uncertainty set $K$ for the drift rate $m$ simplifies to $m \in [m_0 - k\sigma, m_0 + k\sigma]$.

We want to investigate what price the agent will attribute to a derivative that depends both on financial and insurance risk and has payoff $f(t + \Delta t, x(t + \Delta t), y(t + \Delta t))$ at time $t + \Delta t$. Like in Section 3.2, we assume that we can apply a Taylor expansion in the $x$ and $y$ direction up to order $\Delta t$. We then obtain the following expression (where terms of higher order than $\Delta t$ are omitted):

$$E_t[f(t + \Delta t, x(t + \Delta t), y(t + \Delta t)) = f_1 + (f_x' \mu + \frac{1}{2} \text{tr}(f_{xx} \Sigma)) \Delta t, \quad (5.5)$$

with $f_1 := f(t + \Delta t, x(t), y(t))$ and $f_x$ denoting the vector of first order derivatives of $f(t + \Delta t, x(t), y(t))$ with respect to $x(t), y(t)$, and $f_{xx}$ denoting the matrix of second order derivatives.

We furthermore assume that the agent can hedge the financial risk by investing an amount $\theta$ in the share price, but cannot trade in the insurance asset. Hence, the robust rational agent solves the following optimisation problem

$$\max_{\theta} \min_{\mu \in \mathcal{K}} e^{-r\Delta t}(f_1 + (f_x' \mu + \Theta(e_1' \mu - r + \frac{1}{2} \sigma^2) + \frac{1}{2} \text{tr}(f_{xx} \Sigma)) \Delta t), \quad (5.6)$$

where $e_1$ denotes the vector $(1, 0)'$.

We can simplify to an equivalent optimisation problem by substituting $\mu = \mu_0 + \epsilon$ and removing the terms and scaling factors in the objective function that do not depend on the decision variables. This then leads to

$$\max_{\theta} \min_{\epsilon} \epsilon' q + \epsilon' (f_x + \theta e_1)$$

s.t. $\epsilon' \Sigma^{-1} \epsilon \leq k^2. \quad (5.7)$

with $q = (e_1' \mu_0 - r + \frac{1}{2} \sigma^2)$.

The optimal “worst” action of mother nature for the inner minimisation problem (conditional on $\theta$) is to select a disturbance $\epsilon$ of the drift vector $\mu$ along the direction $-\Sigma(f_x + \theta e_1)$, which is then scaled such that the length of $\epsilon^*$ satisfies $\epsilon'^* \Sigma^{-1} \epsilon = k^2$. This solution can be expressed as

$$\epsilon^* := -\left(\frac{k}{\sqrt{(f_x + \theta e_1)' \Sigma (f_x + \theta e_1)}}\right) \Sigma (f_x + \theta e_1). \quad (5.8)$$
If we substitute this optimal value for $\varepsilon^*$ into (5.7), we obtain the reduced optimisation problem

$$
\max_{\theta} \theta q - k\sqrt{(f_x + \theta e_1)' \Sigma (f_x + \theta e_1)}.
$$

(5.9)

Let us take a closer look at this optimisation problem. The second term in the objective function is always negative, and can be seen as a "penalty term" that is given by $k$ times the $\Sigma$-weighted length of the derivative of the position $(f + \theta)$ to a change in $(x, y)$. In other words, the penalty term measures the exposure of the position $(f + \theta)$ to the uncertainty in the expected return $\mu$. The first term is the expected excess return $\theta q$ that the agent can make by investing in the financial market. Note, that this term does no longer contains any uncertainty about $m$, as this is all reflected in the second term. The optimisation problem of the agent now boils down to finding the optimal trade-off between earning more expected return versus the increased exposure to the uncertainty surrounding the expected return.

The first order condition for the optimum of (5.9) is

$$
q = k \frac{e_1' \Sigma (f_x + \theta e_1)}{(f_x + \theta e_1)' \Sigma (f_x + \theta e_1)}.
$$

(5.10)

If we square both sides, we obtain an equation that is quadratic in $\theta$, and has two solutions. However, by squaring (5.10) we introduced a "false root" (i.e. the solution to the optimisation problem: $\min_{\theta} \theta q + k\sqrt{-\gamma}$). Therefore, the unique solution to (5.10) is given by

$$
\theta^* := -\left(f_x + \frac{b \rho}{\sigma} f_y \right) + \frac{q/\sigma}{\sqrt{k^2 - (q/\sigma)^2}} \frac{b \sqrt{1 - \rho^2}}{\sigma} f_y |f_y|.
$$

(5.11)

**Switch of notation: back to scalar expressions $f_x$ and $f_y$!** There are several interesting things to note about this solution. First, the solution $\theta^*$ is only well-defined for $(q/\sigma)^2 < k^2$. This corresponds exactly to the necessary condition (3.8) for a well-defined solution from Section 3.3. Stated differently, if the agent is confident that even in the worst case a positive excess return can be made by investing in the financial market (i.e. when $(q/\sigma)^2 > k^2$), then the agent will try to invest a massive amount in the financial market and has a confident expectation to get very rich.

The second interesting thing to note is that the optimal hedge position consists of two parts: the hedge portfolio $-(f_x + \frac{b \rho}{\sigma} f_y)$ and a "speculative" portfolio that is determined by the product of the residual unhedgeable risk $b \sqrt{1 - \rho^2} \sqrt{k^2 - (q/\sigma)^2}$ and the "market confidence factor" $(q/\sigma)/\sqrt{k^2 - (q/\sigma)^2}$. Note that the quantity
$q/\sigma$ is the market price of risk. The market confidence factor shoots to infinity if $q/\sigma$ approaches $k$. For small values of $q/\sigma$, the market confidence factor is approximately equal to $q/(k\sigma)$, and the speculative investment is then approximately proportional to the market price of risk $q/\sigma$ scaled down by a factor of $k$.

5.3 Agent’s Valuation

What is now the value of the optimisation problem (5.6) for the optimal policy pair $(\theta^*, \mu_0 + \epsilon^*)$? If we substitute the optimal values from (5.8) and (5.11) and simplify, we get

$$e^{-r\Delta t}(f_1 + (f_x(r - \frac{1}{2}\sigma^2) + f_y a^* + \frac{1}{2}\sigma^2 f_{xx} + \rho \sigma b f_{xy} + \frac{1}{2}b^2 f_{yy})\Delta t),$$

(5.12)

where the drift term $a^*$ for the insurance process is given by

$$a^* = a_0 - q\frac{\rho b}{\sigma} + b\sqrt{1 - \rho^2} \cdot \begin{cases} -\sqrt{k^2 - (q/\sigma)^2} & \text{for } f_y > 0, \\ +\sqrt{k^2 - (q/\sigma)^2} & \text{for } f_y < 0. \end{cases}$$

(5.13)

The expressions for $a^*$ may look a bit complicated, but we can give some very nice interpretations for the expressions. The first interpretation is a geometric interpretation. Recall that the uncertainty set $\mathcal{K}$ is an ellipsoid centred around $\mu_0$. Because the financial component $x$ of the risk vector is perfectly replicated, this means that the uncertainty regarding the mean of the financial risk is eliminated, and is replaced by the risk-free return $(r - \frac{1}{2}\sigma^2)$. The uncertainty for the mean of the insurance process $y$ is now confined to the intersection of uncertainty set $\mathcal{K}$ and the line $m = (r - \frac{1}{2}\sigma^2)$. The intersection of a line and an ellipsoid has two solutions: exactly the two solutions given in (5.13).

We have illustrated this geometric interpretation in Figure 3, where we have used the following parameters: $m_0 = \{4\%, 7\%, 10\%\}$, $r = 4\% + \frac{1}{2}\sigma^2$, $a_0 = 0$, $\sigma = 0.15$, $b = 1$, $\rho = 0.75$ and $k = 2/\sqrt{25} = 0.4$. The three different values of $m_0$ lead to $q = \{0\%, 3\%, 6\%\}$, and these three cases are illustrated in the subfigures 3a, 3b and 3c. The uncertainty set $\mathcal{K}$ is given by the interior of the ellipse, and the point estimate $\mu_0$ is given by the point in the centre. The line $m = (r - \frac{1}{2}\sigma^2)$ is the vertical dotted line, and the intersection with the ellipse gives the two solutions for $a^*$. Figure 3a illustrates the case $q = 0$. In this case we find $a^* = \ldots$
Figure 3: Confidence interval for $\mu$ for different values of $q$. 

Fig. 3a: $q = 0\%$

Fig. 3b: $q = 3\%$

Fig. 3c: $q = 0.06\%$
\[a_0 \pm kb \sqrt{1 - \rho^2}\], which is the “naive” confidence interval for \(a\) equal to the point estimate \(a_0\) plus/minus \(k\) times the unhedgeable insurance risk \(b \sqrt{1 - \rho^2}\). The other sub-figures illustrate that for larger values of \(q\), the ellipse moves to the right. This is a reflection of the fact that higher values of \(q\) correspond to higher point-estimates of \(m_0\). When the ellipse moves to the right, the points \(a^*\) move down (due to the correlation term \(-q \rho b / \sigma\)), and the points move closer together (due to the factor \(\sqrt{k^2 - (q / \sigma)^2}\)). Figure 3c illustrates the largest allowed value for \(q\). For larger values of \(q\), the ellipse no longer intersects the line \(m = (r - \frac{1}{2} \sigma^2)\), and the optimisation (5.6) has no longer a finite solution.

The second interpretation for \(a^*\) is an economic interpretation. Recall that the formula for the optimal hedge given in (5.11) consists of two parts: a hedge portfolio and a “speculative” portfolio that depends on \(q\). Suppose that the agent would only choose the hedge portfolio \(\tilde{\theta} := -(f_x + b \rho / \sigma f_y)\). If we substitute \(\tilde{\theta}\) into (5.6), we obtain an expression very similar to (5.12), except that the drift term for the insurance process would be given by \(\tilde{a} = a_0 - q \rho b / \sigma \pm kb \sqrt{1 - \rho^2}\). Therefore, by including the speculative portfolio into the optimal hedge, the agent can finance part of uncertainty in \(a^*\) by exploiting the expected excess return on equities. This then results in the optimal drift term \(a^*\), where the residual unhedgeable insurance risk \(kb \sqrt{1 - \rho^2}\) is shrunk by an additional factor \(\sqrt{k^2 - (q / \sigma)^2}\).

If we now employ a similar Taylor-expansion and take the limit for \(\Delta t \downarrow 0\) as in “Pelsser (2010)”, we find the following pde for the robust pricing operator \(\pi^r(t, x, y)\):

\[
\pi_t^r + (r - \frac{1}{2} \sigma^2) \pi_x^r + a^* \pi_y^r + \frac{1}{2} \sigma^2 \pi_{xx}^r + \rho \sigma b \pi_{xy}^r + \frac{1}{2} b^2 \pi_{yy}^r - r \pi^r = 0, \tag{5.14}
\]

where subscripts denote partial derivatives.

We clearly see in the pde the impact of the market-consistency, because the \(x\) process has the arbitrage-free drift-term \((r - \frac{1}{2} \sigma^2)\).

Note that the insurance process is evaluated using a new drift term \(a^*\) given by (5.13), which reflects the impact of the “worst case” estimate of the robust agent of the drift of the insurance process \(y\). In fact, this new drift term makes the pde non-linear.

Discuss special case \(q = 0\). Back to “Pelsser (2010)” conditional expectation operator, without using conditional expectations!

6 Applications

* Pricing long-dated cash flow with interest rate risk.
* Pricing LT cash flow with equity & int.rate risk.
* Pricing cash flows with mortality risk.
7 Summary and Conclusions
References


